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A static memory sparse spectral method for time-fractional PDEs

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Talk structure

1. Motivation from photoacoustic ultrasound imaging

2. The Yuan-Agrawal method for Caputo derivatives

3. A static memory, sparse and recursive solver

4. Numerical experiments

1. Motivation from photoacoustic ultrasound imaging

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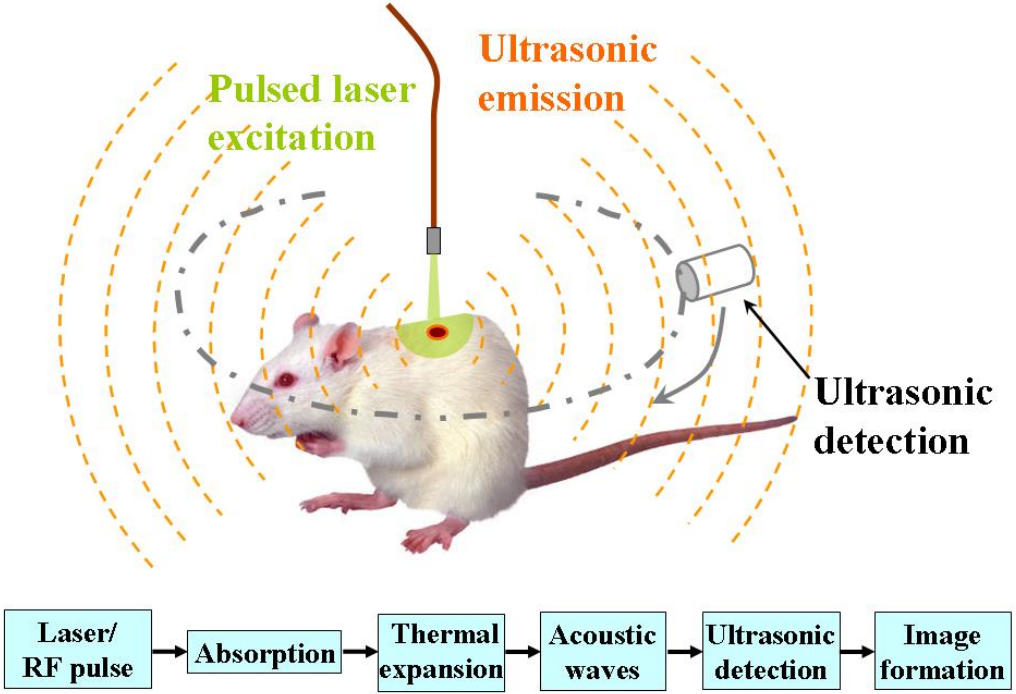

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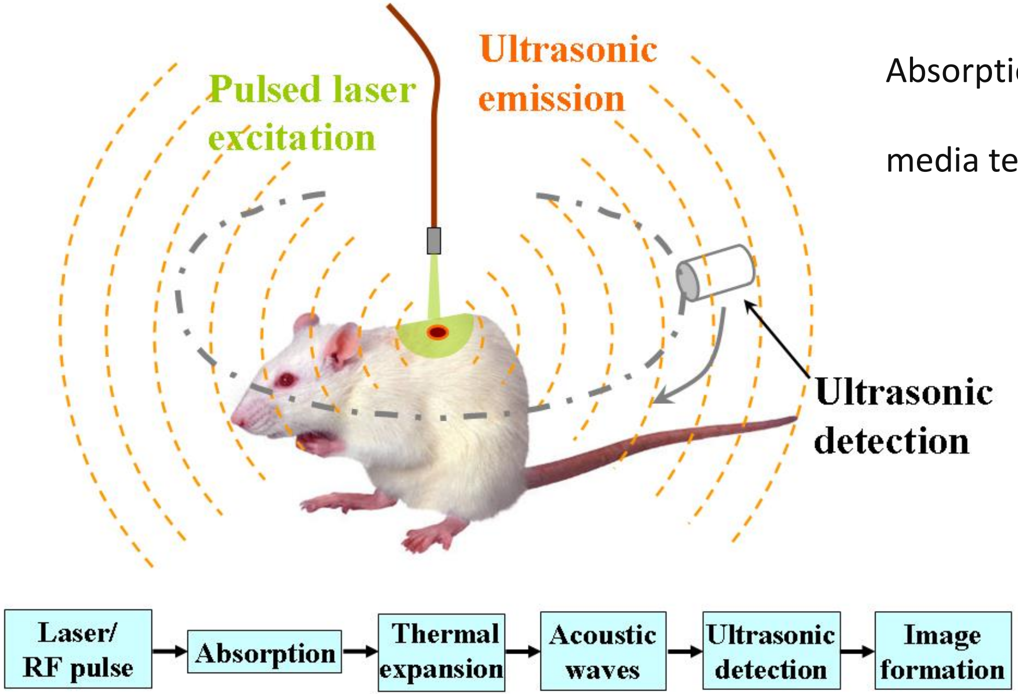
1. Motivation from photoacoustic ultrasound imaging



Absorption of compressional and shear waves in viscoelastic media tend to follow a frequency power law[1].

$$\mathbf{u} = \nabla \phi + \nabla \times \boldsymbol{\Psi}$$

$$0 = \nabla \left(\frac{\partial^2 \phi}{\partial t^2} - c_p^2 \nabla^2 \phi - \tau_p c_p^2 \frac{\partial^{y-1}}{\partial t^{y-1}} \nabla^2 \phi \right) + \nabla \times \left(\frac{\partial^2 \boldsymbol{\Psi}}{\partial t^2} - c_s^2 \nabla^2 \boldsymbol{\Psi} - \tau_s c_s^2 \frac{\partial^{y-1}}{\partial t^{y-1}} \nabla^2 \boldsymbol{\Psi} \right).$$

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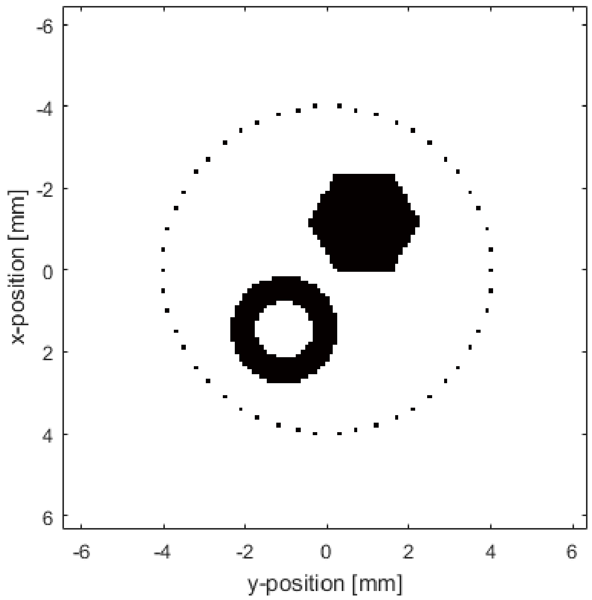
[1] Treeby & Cox 2014, The Journal of the Acoustical Society of America

1. Motivation from photoacoustic ultrasound imaging



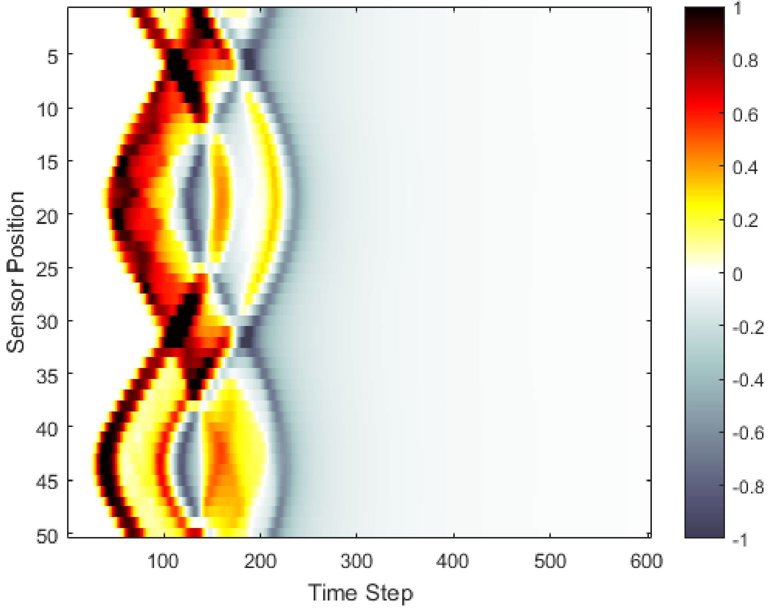
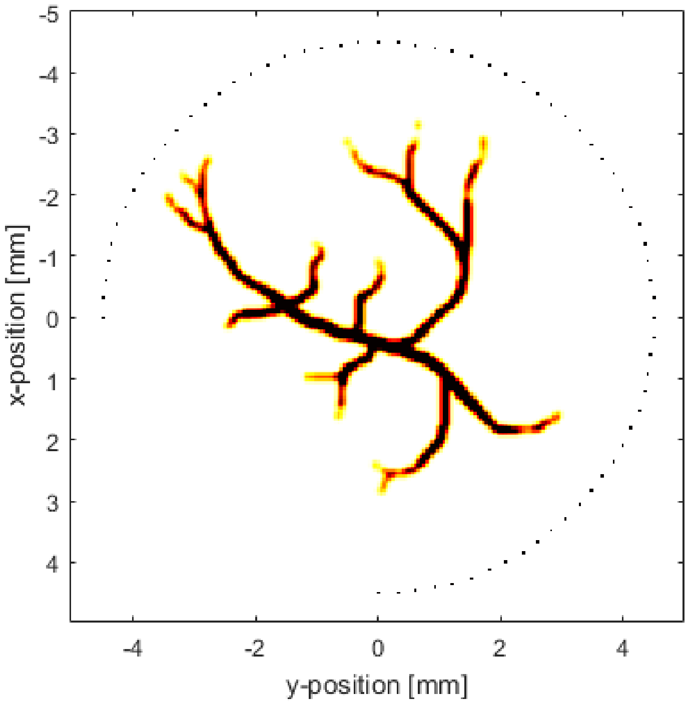


Image Credit: k-wave MATLAB package documentation
http://www.k-wave.org/documentation/example_ivp_loading_external_image.php

1. Motivation from photoacoustic ultrasound imaging



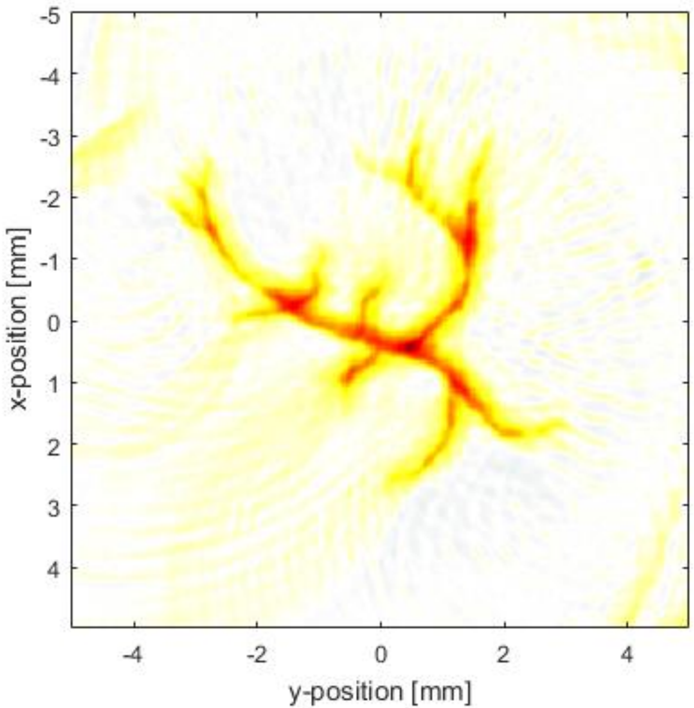


Image Credit: k-wave MATLAB package documentation
http://www.k-wave.org/documentation/example_pr_2D_tr_circular_sensor.php

2. The Yuan-Agrawal method for Caputo derivatives

Definition (Caputo derivative):

$$\frac{\partial^\alpha}{\partial t^\alpha} f(t) = D_C^\alpha f(t) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^t (t-s)^{(\lceil\alpha\rceil-\alpha-1)} f^{(\lceil\alpha\rceil)}(s)\,\mathrm{d}s,$$

Inherent challenge: Non-local in time → Memory accumulation

Treeby & Cox's solution:
Transform the *time*-fractional into a *space*-fractional problem.

$$\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\phi - \nabla^2\phi - \tau\frac{\partial^{y-1}}{\partial t^{y-1}}\nabla^2\phi = 0.$$

$$\downarrow$$

$$\frac{1}{c_0^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2\phi - \tau_1\left(-\nabla^2\right)^{y/2}\frac{\partial}{\partial t}\phi - \tau_2\left(-\nabla^2\right)^{(y+1)/2}\phi = 0.$$

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$$\downarrow$$

$$\mathcal{F}_{x,t}\left\{\frac{\partial^y g(x,t)}{\partial t^y}\right\} = (-i\omega)^y G(k,\omega).$$

$$\omega\tau \ll 1$$

$$\frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi - \tau_1(-\nabla^2)^{y/2}\frac{\partial}{\partial t}\phi - \tau_2(-\nabla^2)^{(y+1)/2}\phi = 0.$$

THEOREM 3.1 (Generalized Yuan–Agrawal–Caputo derivative). *Let* $\alpha > 0$, $\alpha \notin \mathbb{N}$ *and* $f \in C^{\lceil \alpha \rceil}[0,T]$. *Then the Caputo fractional derivative of* f *can be expressed as*

$$\frac{\partial^\alpha}{\partial t^\alpha} f(t) = \int_0^\infty \phi(w,t)\mathrm{d}w,$$

where the function $\phi_f : (0,\infty) \times [0,T] \to \mathbb{R}$ *is defined by*

$$\phi_f(w,t) := \frac{(-1)^{\lfloor \alpha \rfloor} 2\sin(\pi\alpha)}{\pi} w^{2\alpha - 2\lceil \alpha \rceil + 1} \int_0^t e^{-w^2(t-\tau)} \frac{\partial^{\lceil \alpha \rceil}}{\partial \tau^{\lceil \alpha \rceil}} f(\tau)\mathrm{d}\tau.$$

Furthermore, for fixed $w > 0$ *the function* $\phi_f(w,t)$ *satisfies the (non-fractional) differential equation*

$$\frac{\partial}{\partial t}\phi_f(w,t) = -w^2\phi_f(w,t) + \frac{(-1)^{\lfloor \alpha \rfloor} 2\sin(\pi\alpha)}{\pi} w^{2\alpha - 2\lceil \alpha \rceil + 1} \frac{\partial^{\lceil \alpha \rceil}}{\partial t^{\lceil \alpha \rceil}} f(t), \tag{3.1}$$

with initial condition $\phi(w,0) = 0$.

[1] Yuan & Agrawal, 2002, J. Vib. Acoust.

Criticism of the Yuan-Agrawal method:
Gauss-Laguerre quadrature tends to work very poorly for these problems and requires prohibitively large number of nodes.

A solution to this was proposed by Diethelm[1] (and Birk & Song[2]):

$$\int_0^\infty \phi_f(w,t)\mathrm{d}w = \int_{-1}^1 (1-\kappa)^{\bar{\alpha}}(1+\kappa)^{-\bar{\alpha}}\bar{\phi}_f(\kappa,t)\mathrm{d}\kappa,$$

$$\bar{\phi}_f(\kappa,t) := 2(1-\kappa)^{-\bar{\alpha}}(1+\kappa)^{\bar{\alpha}-2}\phi_f\left(\frac{1-\kappa}{1+\kappa},t\right).$$

[1] Diethelm 2008, Numerical Algorithms
[2] Birk & Song 2010, Comput. Mech

3. A static memory, sparse and recursive solver

Restate Yuan-Agrawal method in recursive form:

$$\frac{\partial^\alpha}{\partial t^\alpha} f(t) \approx \sum_{j=1}^{L} A_j \int_0^t e^{-s_j^2(t-\tau)} \frac{\partial^{\lceil\alpha\rceil}}{\partial \tau^{\lceil\alpha\rceil}} f(\tau)\mathrm{d}\tau = \sum_{j=1}^{L} A_j \psi_j(t),$$

$$\psi_j(t) := \int_0^t e^{-s_j^2(t-\tau)} \frac{\partial^{\lceil\alpha\rceil}}{\partial \tau^{\lceil\alpha\rceil}} f(\tau)\mathrm{d}\tau.$$

$$\downarrow$$

$$\psi_j(t) = e^{-s_j^2 \Delta t} \psi_j(t-\Delta t) + \int_{t-\Delta t}^{t} e^{-s_j^2(t-\tau)} \frac{\partial^{\lceil\alpha\rceil}}{\partial \tau^{\lceil\alpha\rceil}} f(\tau)\mathrm{d}\tau.$$

Resolve spatial dependence in multivariate orthogonal polynomials:

$$f(n\Delta t, \mathbf{x}) = \mathbf{P}(\mathbf{x})\boldsymbol{f}(n\Delta t),$$

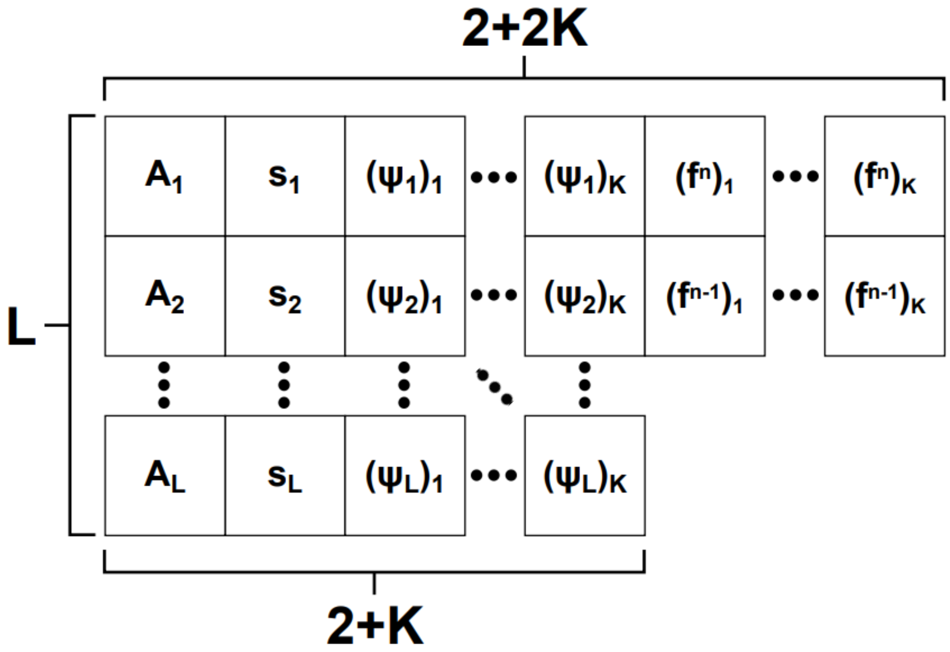
$$\psi_j(n\Delta t, \mathbf{x}) = \mathbf{P}(\mathbf{x})\boldsymbol{\psi}_j(n\Delta t).$$

We obtain:

$$\left(\frac{\partial^\alpha f}{\partial t^\alpha}\right)(n\Delta t, \mathbf{x}) \approx \mathbf{P}(\mathbf{x})\sum_{j=1}^{L} A_j\left(e^{-s_j^2\Delta t}\boldsymbol{\psi}_j^{n-1} + \frac{(1-e^{-s_j^2\Delta t})}{s_j^2\Delta t}\left(\boldsymbol{f}^n - \boldsymbol{f}^{n-1}\right)\right),$$

$$\psi(n\Delta t, \mathbf{x}) = \mathbf{P}(\mathbf{x})\boldsymbol{\psi}(n\Delta t) \approx \mathbf{P}(\mathbf{x})\boldsymbol{\psi}_j^n = \mathbf{P}(\mathbf{x})\left(e^{-s_j^2\Delta t}\boldsymbol{\psi}_j^{n-1} + \frac{(1-e^{-s_j^2\Delta t})}{s_j^2}\frac{\boldsymbol{f}^n - \boldsymbol{f}^{n-1}}{\Delta t}\right).$$

The memory requirements of computing a Caputo derivative:

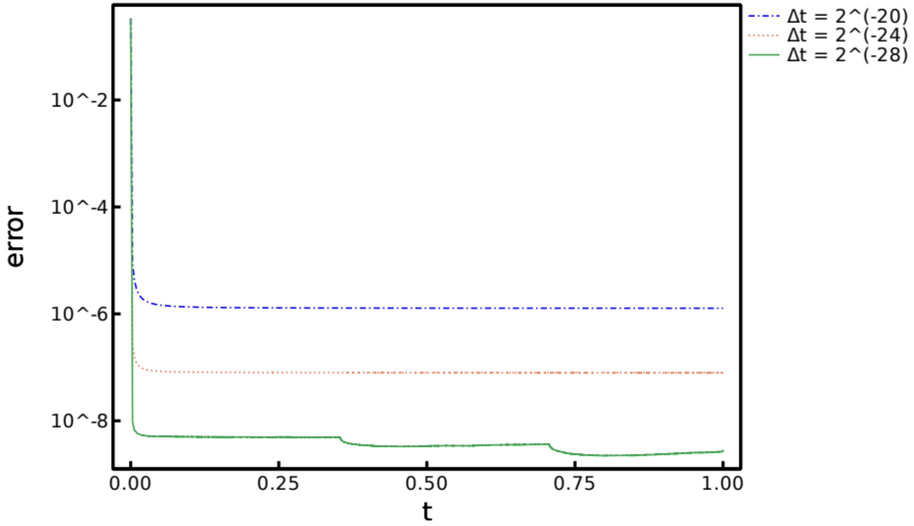


L(2+K)+2K, where

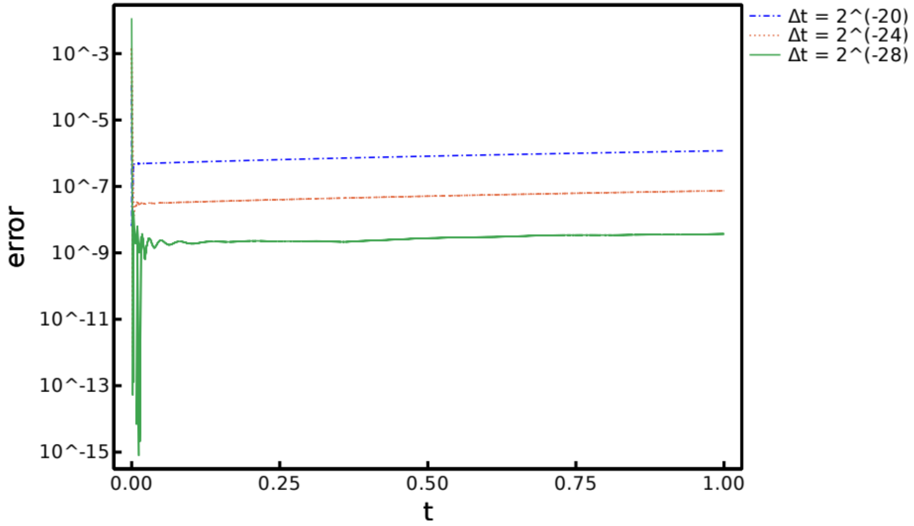
L … number of quadrature pts.

K … degree of polynomial approx.

4. Numerical experiments

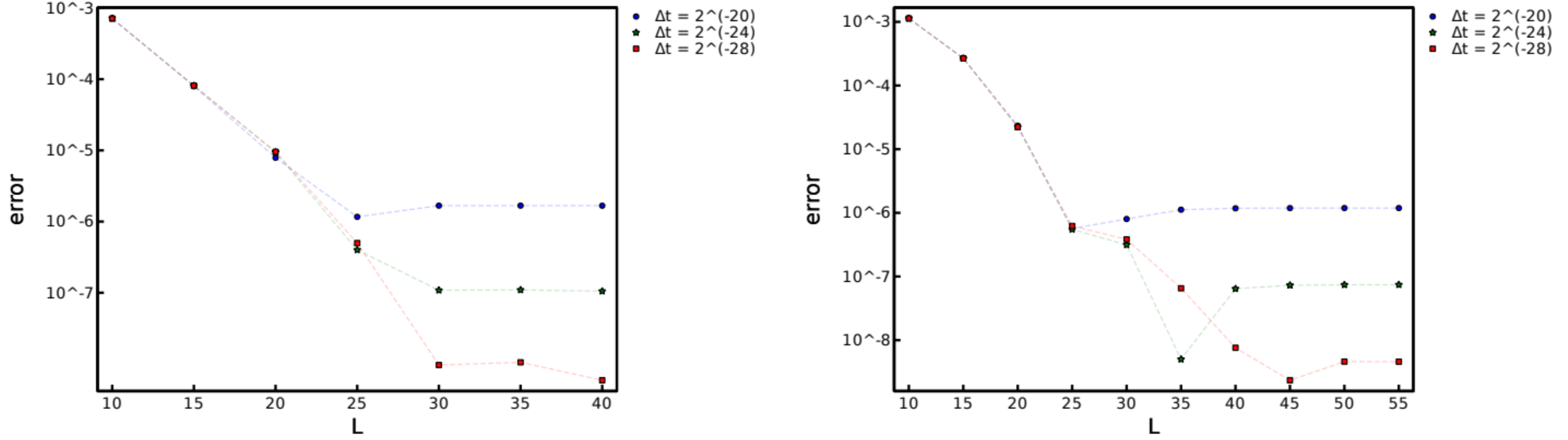


(a) $f(t) = t^2,\ L = 65,\ \alpha = \frac{2}{3}$



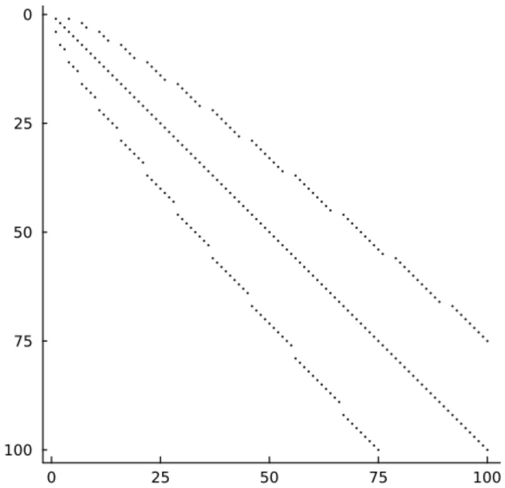
(b) $f(t) = e^t,\ L = 60,\ \alpha = \frac{1}{2}$

$$\left| \left(\frac{\partial^\alpha}{\partial t^\alpha} f \right) (T, \mathbf{x}) - \mathbf{P}(\mathbf{x}) \mathcal{P}_K \sum_{j=1}^{L} A_j \boldsymbol{\psi}_j^N \right| \leq \frac{M(\mathbf{x})}{(2L)!} + C(\mathbf{x}) L T \Delta t + \mathrm{err}_{\mathbf{P},K} \left(\sum_{j=1}^{L} A_j \psi_j(T, \mathbf{x}) \right).$$



(a) $f(t) = t^2$, $\alpha = \frac{1}{4}$ (b) $f(t) = e^t$, $\alpha = \frac{1}{2}$

Discretize our equation of interest on the unit disk:

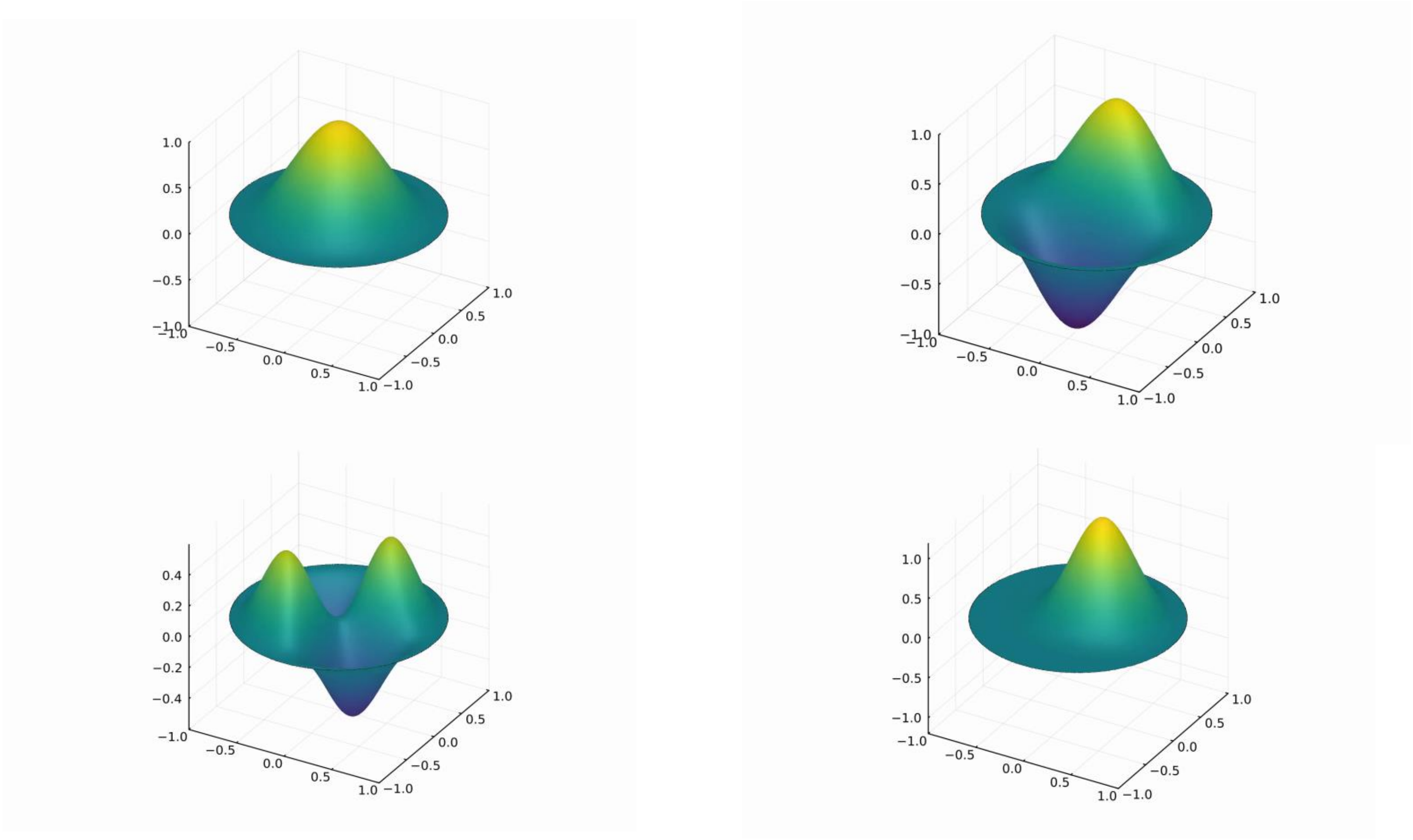


$$\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}f - \Delta f + \tau\frac{\partial^\alpha}{\partial t^\alpha}f = 0,$$

$$\downarrow$$

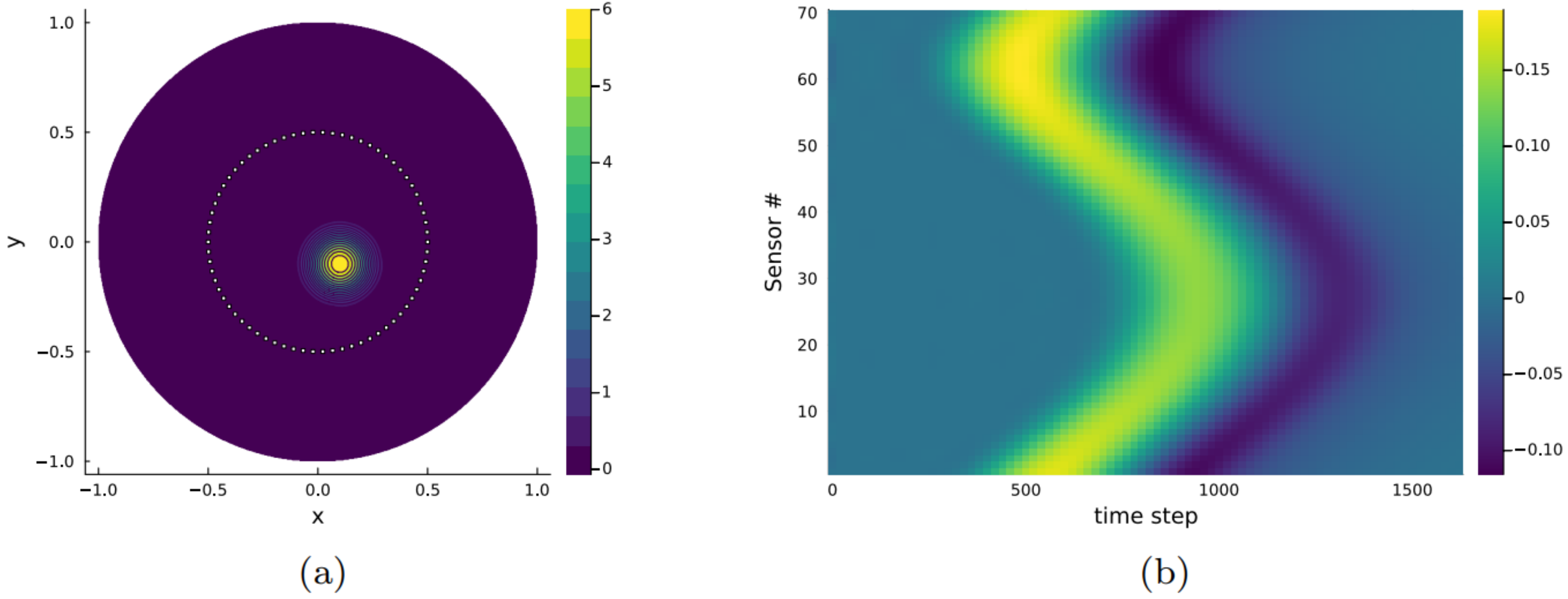
$$\mathbf{Z}^{(1)}(r,\theta)\left(\left(\frac{1}{c_0^2\Delta t} + \tau\left(\sum_{j=1}^{L} A_j \frac{1-e^{-s_j^2\Delta t}}{s_j^2}\right)\right)\mathcal{C} - (\Delta t)\Delta\right)\boldsymbol{f}^n =$$

$$\mathbf{Z}^{(1)}(r,\theta)\mathcal{C}\left(\left(\frac{2}{c_0^2\Delta t} + \tau\sum_{j=1}^{L} A_j \frac{1-e^{-s_j^2\Delta t}}{s_j^2}\right)\boldsymbol{f}^{n-1} - \frac{\boldsymbol{f}^{n-2}}{c_0^2\Delta t} - \tau\sum_{j=1}^{L} A_j e^{-s_j^2\Delta t}\boldsymbol{\psi}_j^{n-1}\right),$$



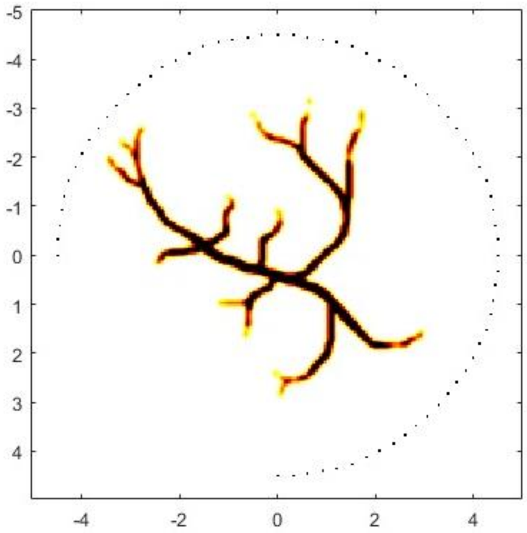
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
−1.0
−0.5
0.0
0.5
1.0
0.4
0.2
0.0
−0.2
−0.4

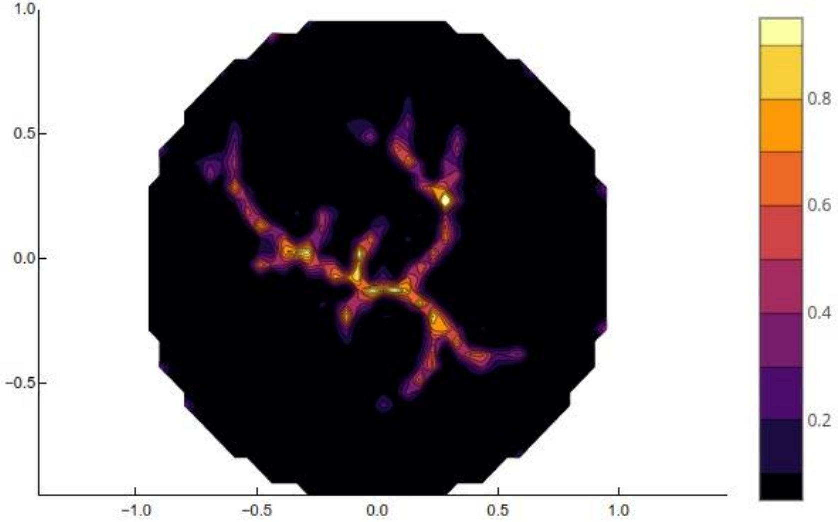
Modeling circular sensor arrays:



Next goals:

- Fully realized image reconstruction
- Comparison with fractional Laplacian methods
- Combine with spectral element methods on annular domains





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