A sparse spectral method for power law equilibrium measures and Riesz potentials

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Equilibrium measures

We study the equilibrium states of the purely attractive repulsive classical N particle system in d dimensions:

$$\frac{d^2x}{dt^2} = -\frac{1}{N}\sum_{j\neq i}\nabla K(|x-y|),$$

with power law interaction kernel

$$K(|x-y|) = \frac{|x-y|^\alpha}{\alpha} - \frac{|x-y|^\beta}{\beta}.$$

The continuous limit $N \to \infty$ is an aggregation equation whose equilibrium states are among the minimizers of

$$\frac{1}{\alpha}\int_{\mathrm{supp}(\rho)} |x-y|^\alpha \rho(y)\mathrm{d}y - \frac{1}{\beta}\int_{\mathrm{supp}(\rho)} |x-y|^\beta \rho(y)\mathrm{d}y = E.$$

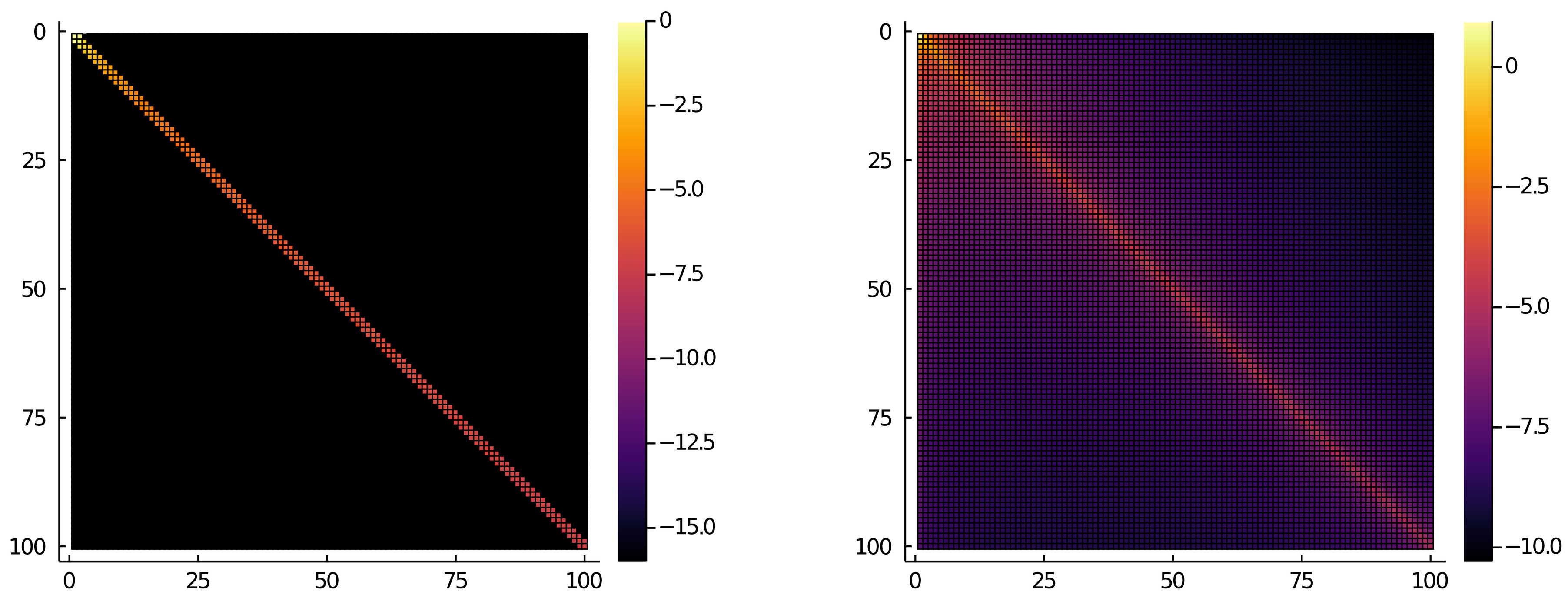
Animal swarms, cellular motion in a petri dish as well as classical physical particle systems like charged dust particulates are all applications in which equilibrium measure problems naturally appear.

Banded Riesz potentials

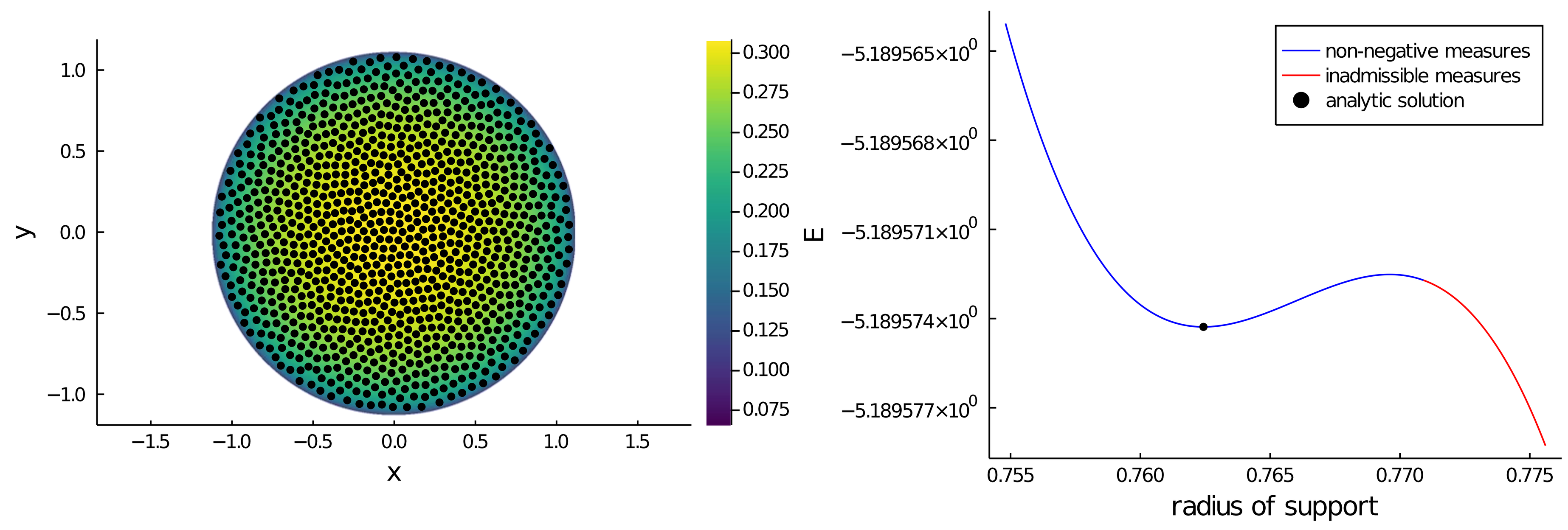
The radial symmetry of the problem suggests the use of symmetric Jacobi polynomials on ball domains. We can show that these orthogonal polynomials behave nicely with respect to Riesz (= power law) potentials:

$$\int_{B_1} |x-y|^\gamma (1-y^2)^{m-\frac{\alpha+d}{2}} P_n^{(m-\frac{\alpha-d}{2},\frac{d-2}{2})}(y)\mathrm{d}y$$
$$= const(\gamma,\alpha,d)\ {}_2F_1\left(n-\frac{\gamma}{2}, -m-n-\frac{\alpha-\gamma}{2}, \frac{d}{2}; |x|^2\right).$$

The above ${}_2F_1$ function is a polynomial if $\gamma = \alpha$. The repulsive β-part is then approximated in the same basis, yielding a band-dominant matrix. The entries of the matrices are computed via ${}_2F_1$ recurrence relationships.



Left: Banded matrix for attractive part for $\alpha = 1.3$ in dimension $d = 2$.
Right: Band-dominant matrix for repulsive part in the same basis for $\beta = \frac{1}{\pi}$.
(Note: Logarithmic legend, showing order of magnitude of absolute value of elements)



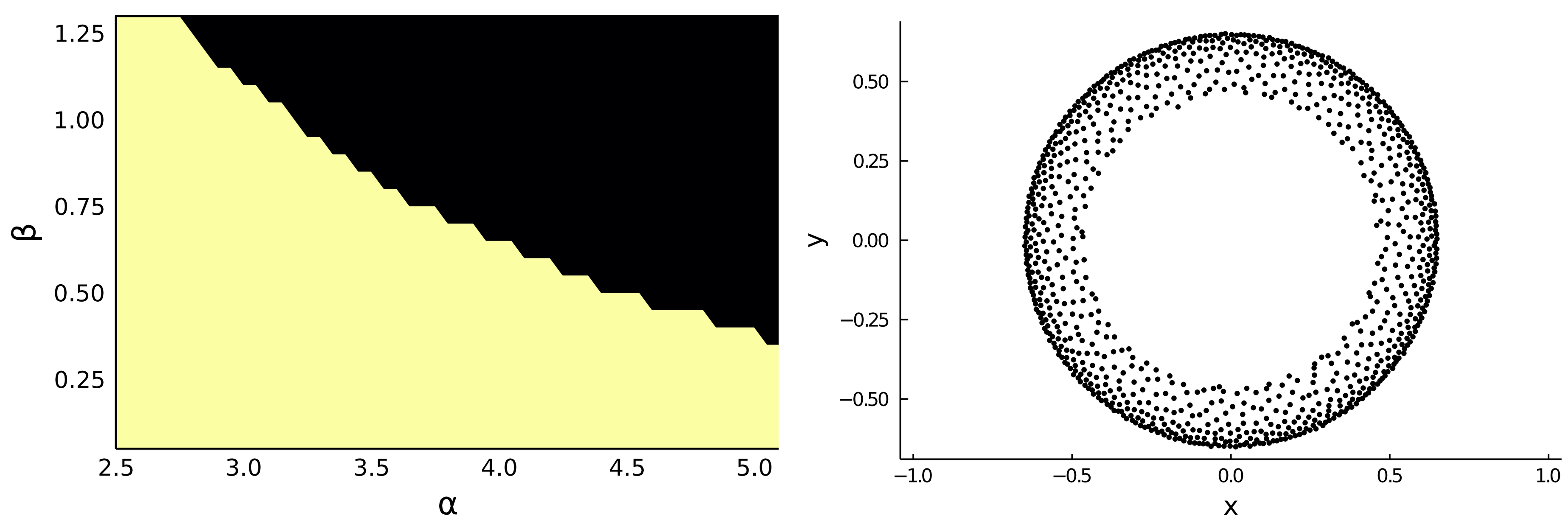
Left: Combined particle simulation and computed solution for $\alpha = 2,\ \beta = -0.44,\ d = 2$.
Right: Plot of energy as a function of the radius of the ball support for $\alpha = 4,\ \beta = \frac{1}{\pi},\ d = 2$.

Uniqueness of solutions

Guessing an initial support radius we can use constrained optimization techniques to find minimizers.

Our method exactly reproduces solutions for the few special cases with analytically known results.

For generic powers within 'low' repulsive domains without analytic results, our method also suggests that uniqueness of non-negative minimizers holds.



Left: Contour plot of regions with ball support (light) vs. gap around origin (dark) for $d = 2$.
Right: Example of post-gap formation equilibrium state ($\alpha = 4.2, \beta = 0.85, d = 2$) computed via particle simulation.

Exploring gap formation

For high repulsive powers our method allows exploration of the poorly understood gap formation boundary after which the support collapses to spherical shells or rings.

Where such gap formation occurs, all obtained measures fail to be non-negative. Methods to compute the spherical shell solutions with spectral methods are work in progress.

References

- *Computation of Power Law Equilibrium Measures on Balls of Arbitrary Dimension, T. S. Gutleb , J. A. Carrillo , S. Olver, 2021, arXiv:2109.00843*
- *Computing Equilibrium Measures with Power Law Kernels, T. S. Gutleb , J. A. Carrillo , S. Olver. (to appear in Math. Comp., 2022)*